Riemannian Structure on Lie Groups

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1 Introduction

Lie groups are special objects which are endowed simultaneously with an algebraic structure and a geometric structure. Unifying the two structures are conditions we impose that make smooth and discrete structure compatible in some manner. Lie groups are of significant interest because they describe continuous symmetries which appear in areas from theoretical physics to differential equations. As such, it is natural that we would like to study geometric objects which interact well with the algebraic structure. To begin, a Lie group is a smooth manifold with a group structure. Our compatibility requirement is that the group operation and inversion must be smooth maps with respect to the manifold structure. Conversely, geometric objects which respect the group operation are referred to as invariant. From the invariant vector fields on a Lie group, we can define a vector space equipped with a Lie bracket which is known as a Lie algebra. We can naturally extend this notion into the realm of Riemannian structures by studying metrics, connections, and curvatures which respect the group structure.

2 Lie groups and Lie algebras

We begin with the definition of a Lie group. Throughout we will let G be a group with identity e and multiplication $*$. A group G is an **r-parameter Lie group** if it also has the structure of an r-dimensional smooth manifold such that the group operation $(g, h) \mapsto g * h$ and the inversion map $i : g \mapsto g^{-1}$ are smooth maps with respect to the manifold structure.

It is important to keep in mind that Lie groups generally describe continuous symmetries when picturing a Lie group. Therefore, we will reference important symmetry groups when giving examples of Lie groups.

Examples

1. Consider the circle described as $SO(2)$, which is the collection of transformation on the circle and equals the set of 2×2 matrices such that the determinant

is 1. This is a 1−parameter Lie group with group action being composition of transformations. More generally, we can define $SO(n)$, the special orthogonal group

2. The special unitary group $SU(n)$ is the collection of unitary matrices with determinant 1. This group is a generalization of $SO(n)$ to complex variables.

It naturally follows that a **Lie group homomorphism** $\phi : G \to H$ between two Lie groups is a group homomorphism that is also a smooth map between manifolds.

As we are dealing with manifolds, it is natural that we want to describe them in the language of tangent spaces. Indeed, we have tangent spaces, however, we wish to study vector fields that somehow also respect the group structure of our Lie groups. This is where the notion of invariance first appears. To study this we first introduce the left (or right) multiplication map. The left (or right) multiplication map of an element $g \in G$ is defined

$$L_g(h) = g * h \text{ (or } R_g(h) = h * g)$$

Henceforth, we will only refer to left multiplication as everything is parallel for left multiplication. Note that these both define smooth maps since group multiplication is required to be smooth. In fact, these maps are diffeomorphisms with inverse $dL_{g^{-1}} = (dL_g)^{-1}$. Therefore, the differential dL_g is well defined. A vector field V on G is said to be or left-invariant if $dL_g(V_h) = V_{g*h}$ (note the abuse of notation with respect to dL_g as this should be $(dL_g)_h$) for all $g, h \in G$. From here, we define the **Lie algebra $\mathfrak{g}$ of** G to be the vector space of left-invariant vector fields on G. Note that since L_g is a diffeomorphism for all g, then dL_g is an isomorphism of vector spaces. In particular, this means that the action of any left-invariant vector field V can be determined from the value it takes at the identity e, or, for any g, $V_g = dL_g(V_e)$. This tells us that $T_gG \cong \mathfrak{g}$ for all $g \in G$.

Examples

1. Let $G = (\mathbb{R}, +)$. Let x be the coordinate on G. Then for $h \in \mathbb{R}$, $R_h(x) = x + h$. Then $dR_h(\frac{\partial}{\partial x}) = \frac{\partial R_h}{\partial x}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$. This implies $\frac{\partial}{\partial x}$ is right invariant, and therefore in the Lie algebra. However, $x\frac{\partial}{\partial x}$ is not right invariant. Therefore, the Lie algebra is $\mathfrak{g} = span_{\mathbb{R}}\{\frac{\partial}{\partial x}\}$.

2. Let $G = GL_n(\mathbb{R})$. This is an n^2−parameter Lie group, so $\mathfrak{g} \cong \mathbb{R}^{n\times n}$ as vector spaces and they are isomorphic as Lie algebras with $\mathbb{R}^{n\times n}$ having the matrix commutator $AB - BA$ as its Lie bracket. Consider $A \in \mathbb{R}^{n\times n}$. Then the vector field induced by it is $v_A|_I = a_{ij}\frac{\partial}{\partial x^{ij}}$. For any $Y \in GL(n)$, we have

$$v_A|_Y = dR_Y(v_A|_I) = \sum_{i,j,m} a_{ij}y_{jm}\frac{\partial}{\partial x^{im}} = v_{AY}|_I$$

One can check that, for the Lie bracket, $[v_A, v_B] = v_{AB-BA}$, i.e., the matrix commutator.

More generally, a Lie algebra is defined to be a vector space equipped with a Lie bracket. However, as the Lie bracket of two left-invariant vector fields will be left-invariant, our notion of a Lie algebra induced from a Lie group aligns with the more general concept. Given a Lie algebra $\mathfrak{g}$ with Lie bracket $[\cdot, \cdot]$, we can define its **structure constants** in the following way. Let $\{e_1, ..., e_n\}$ be a vector space basis for $\mathfrak{g}$. Then its structure constants are the dot products

$$\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle \text{ for } i, j, k = 1, ..., n$$

Clearly, based on this definition, we can see that the structure constants are skew-symmetric in the first two indices.

Finally, we discuss two important maps, the Lie exponential map and the adjoint representation, before introducing Riemannian structure. To define the exponential map, consider the exponentiation of a vector field defined by $\exp(t, V)x = \Psi(t, x)$ where Ψ is the flow generated by V. Then we define the Lie exponential map to be $g(t) = \exp(tV)e$. A property of this map is that $g_t * g_s = g_{t+s}$. In this way, we can define an action on G by $\mathbb{R}$ by a given V, called an infinitesimal generator. Now, we introduce the adjoint representation. This is a map $Ad : G \to Aut(\mathfrak{g})$ defined by $g \mapsto d(C_g)_e$ where C_g is the conjugation map. Therefore, $d(C_g)_e = d(L_g)_{g^{-1}} \circ d(R_{g^{-1}})_e$. More concretely, the map g is sent to can be described by

$$Ad(g)V = \tfrac{d}{dt}(g \exp(tV) g^{-1})|_{t=0}$$

Further, we can define the adjoint map for on $\mathfrak{g}$, ad, by $ad_X(Y) = [X, Y]$. These two maps are related in that $ad = d(Ad)_e$. Note that ad is defined for vector fields. However, recall that every vector in $\mathfrak{g}$ is associated with a left-invariant vector field on G through an isomorphism.

3 Invariant Metrics

Now, moving onto Riemannian structure, we want to know when we can obtain interesting Riemannian structure on Lie groups. Being the basis for Riemannian structure, we will first examine metrics. From Riemannian geometry, a Riemannian metric is a positive definite scalar product $g_p : T_pG \times T_pG \to \mathbb{R}$ defined for each $p \in G$. Now, recalling the adjoint representation from above, in general, a **representation** of a Lie group is a smooth homomorphism $\rho : G \to Aut(V)$ for some finite dimensional vector space V. However, we will here let $V = \mathfrak{g}$. Remembering that metrics are inner products defined on the tangent space of a manifold, we have the follow definition to motivate what it means for a metric to be invariant. For a representation ρ, the inner product $\langle \cdot, \cdot \rangle$ is invariant if $\langle \rho(g)U, \rho(g)V \rangle = \langle U, V \rangle$ for all $U, V \in \mathfrak{g}$ and all $g \in G$.

This brings us to the definition of an invariant metric. A **left-invariant metric** on a Lie group G is a Riemannian metric $\langle \cdot, \cdot \rangle$ such that

$$\langle U, V\rangle_g = \langle (dL_h)_g U, (dL_h)_g V\rangle_{ab}$$

for all $g, h \in G$ and all $U, V \in T_g G$. A metric that is left and right-invariant is called **bi-invariant**.

Proposition 1 There is a bijective correspondence between left-invariant metrics on G and inner products defined on $\mathfrak{g}$.

Proof. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ and let X, Y be vector fields on G. We define a metric by

$$\langle X, Y\rangle_g = \langle (dL_{g^{-1}})_g X, (dL_{g^{-1}})_g Y\rangle$$

On the other hand, given a metric on G and $U, V \in \mathfrak{g}$, then we define an inner product by

$$\langle U, V\rangle = \langle U, V\rangle_e$$

□

Furthermore, any Lie group which admits a left-invariant metric is complete.

If we impose a bi-invariant metric on G, the inversion map is not only by definition smooth, it is also an isometry. Since $R_g = i \circ L_{g^{-1}} \circ i$, an important property of the inversion map is this:

Proposition 2 If G has a left-invariant metric, then G also has a left invariant metric if and only if the inversion map is an isometry.

Proof. Ad_h is induced by the smooth mapping $L_h R_h^{-1} : G \to G$. Since the metric g is left-invariant, we have $L_h^* g = g$. If g is also right-invariant, then $(L_h R_h^{-1})^* g = g$, showing that Ad_h is an isometry. Consequently if we assume Ad_h is an isometry, then clearly we have g is right-invariant since $L_h^* g = g$. □

Furthermore, if we restrict our case of G, we see that

Proposition 3 Let G be a connected Lie group with a left-invariant metric. Then the metric is bi-invariant if and only if ad_X is skew-adjoint for every $X \in \mathfrak{g}$.

Proof. Suppose $h \in G$ is close enough to e such that $h = \exp(X)$ for X close to $0 \in \mathfrak{g}$. Since $Ad_h = Ad_{\exp(x)} = \exp(ad_X)$ and Ad_h is orthogonal if and only if $Ad_g^{-1} = Ad_g^*$, we have $exp(-ad_X) = \exp(ad_X^*)$ if and only if $-ad_X = ad(X)^*$. Since a connected Lie group is generated by a neighborhood of the identity and products of orthogonal transformations are orthogonal, it follows that $-ad_X = ad(X)^*$. So ad_X is skew-adjoint if and only if Ad_h is skew-adjoint, i.e., we have a bi-invariant metric. □

The adjoint representation is of particular importance is the adjoint representation due to the following correspondence:

Proposition 4 (First Criterion) There is a bijective correspondence between bi-invariant metrics on G and $Ad-$invariant inner products on $\mathfrak{g}$, i.e., inner products for which Ad_g is an isometry for all $g \in G$.

Proof. This follows from Proposition 2 and form the fact that Ad_h is induced by $L_h R_h^{-1}$. □

Given the correspondence between inner products we see, the following theorem from representation theory will help us characterize invariant metrics:

Theorem 1 Let G be a compact Lie group. Then for every representation $\rho : G \to Aut(\mathfrak{g})$, there exists an invariant inner product on $\mathfrak{g}$.

Proof. Let $\langle \cdot, \cdot \rangle$ be an inner product. Define $\langle X, Y \rangle' = \int_G \langle Ad_h X, Ad_h Y]inv \rangle dG$, integrating over $h \in G$. Then this metric is bi-invariant:

$$\int_G \langle Ad_h(Ad_k U), Ad_h(Ad_k V) \rangle dG = \int_G \langle Ad_h U, Ad_h V \rangle dG$$

Since any element can be expressed as the product of two elements, this shows the inner product is constant over G. □

Corollary 1.1 There is an invariant inner product on V if and only if $\rho(G)$ is compact. In particular, if G is compact, there exists an invariant inner product on V.

Theorem 2 (Second Criterion) Let G be a Lie group. An inner product on $\mathfrak{g}$ induces a bi-invariant metric on G if and only if $Ad(G)$ is compact.

Therefore, from the above corollary, we can see immediately that every compact Lie group admits a bi-invariant metric. Theorem 2 can be used to show that certain Lie groups do not admit a bi-invariant metric.

Example The Lie group $SE(n)$ is the collection of Euclidean rigid motions on $\mathbb{R}^n$. That is, the collection of rotations and translations. $SE(n)$ does not admit a bi-invariant metric for $n \geq 3$.

Our next criterion for existence of a bi-invariant metric relies on the adjoint representation on $\mathfrak{g}$, $ad = dAd_e$.

Theorem 3 (Third Criterion) Let G be a connected Lie group. An inner product on $\mathfrak{g}$ induces a bi-invariant metric on G if and only if the linear map $ad_U : \mathfrak{g} \to \mathfrak{g}$ is skew-adjoint for all $U \in \mathfrak{g}$, i.e.,

$$\langle ad_U(V), W \rangle = -\langle V, ad_U(W) \rangle \text{ for all } U, V, W \in \mathfrak{g}$$

In terms of Lie brackets: $\langle [U, V], W \rangle = \langle U, [V, W] \rangle$

Proof. Suppose the metric is bi-invariant such that $\langle U, V\rangle = \langle Ad_h U, Ad_h V\rangle$ where $h = \exp(tW)$. Then

$$\langle U, V\rangle = \langle Ad_{\exp(tW)}U, Ad_{\exp(tW)}V\rangle$$

Then, taking $\frac{d}{dt}|_{t=0}$ results in $0 = \langle UW - WU, V\rangle + \langle U, VW - WV\rangle$ where $UW - WU = [U, W]$. The reverse direction is shown by integration. □

Finally, an important result of Milnor is that a connected Lie group admits a bi-invariant metric if and only if it is isomorphic to the cartesian product of a compact group and a Euclidean vector space.

4 Connections

Naturally, for any metric on a Riemannian manifold, we can define the usual extensions of Riemannian structure. That is, we have connections, in particular, the Levi-Civita connection. Given that a Lie group has additional structure compared to a typical Riemannian manifold, we should expect that we can derive more specific information about these constructs.

With a bi-invariant metric or even just a left-invariant metric, we can come up with more specific formulas for the Levi-Civita connection of the given metric, since it is already a given that it exists.

4.1 Left-invariant Case

Suppose $\langle\cdot,\cdot\rangle$ is a left-invariant metric on a Lie group G. Then for $X, Y \in \mathfrak{g}$, we have the $\langle X, Y\rangle_g$ is constant over varying g. So we can define a constant function $\langle X, Y\rangle$. Then, by Koszul's formula, the Levi-Civita connection ∇ with respect to a left-invariant metric:

$$2\langle\nabla_X Y Z\langle = X = (\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle = -\langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle$$

This follows from the above result. However, it only holds for left-invariant vector fields. By rearranging our Lie brackets we come to the formula $2\langle\nabla_X Y, Z\rangle = \langle[X, Y], Z\rangle - \langle[Y, Z], X\rangle + \langle[Z, X], Y\rangle$. Then, for the connection, we arrive at the formula

$$\nabla_X Y = \tfrac{1}{2}([X, Y] - ad_X^* Y - ad_Y^* X)$$

where ad_X^* represents the adjoint operator of $ad_X \in Aut(\mathfrak{g})$.

4.2 Bi-invariant Case

If we specify that we will only consider bi-invariant metrics, then we can obtain an even better formulation. By our Third Criterion, if we have a bi-invariant metric, then $\langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle$. Therefore, when we return to Koszul's formula on Lie groups:

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle$$

Since $\langle [Z,X],Y\rangle = \langle Y,[Z,X]\rangle = \langle [Y,Z],X\rangle$, then the last two terms cancel, leaving only the first term. Therefore, the Levi-Civita connection is simply given by the Lie bracket:

$$\nabla_X Y = \tfrac{1}{2}[X,Y]$$

5 Curvature

We will explore the formula for section curvature here and some generalizations we can make about sectional curvature for certain cases. Recall the the Riemannian curvature tensor is given by $R(X,Y)Z = \nabla_{X,Y}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z$.

5.1 Left-invariant Case

We will return to the left-invariant case and discuss formations for a few different forms of curvature on Lie groups. With respect to our left-invariant metric, i.e., the inner product on $\mathfrak{g}$, we can find an orthonormal basis $e_1,...,e_n$ on $\mathfrak{g}$. Working within a Lie algebra means we are defining an orthonormal basis on the set of left-invariant vector fields. Then, with respect to ∇ and recalling our structure constants a_{ijk} :

$$\nabla_{e_i}e_j = \tfrac{1}{2}(a_{ijk} - a_{jki} + a_{kij})e_k$$

Note that we are using Einstein summation notation over the index k. Irrespective of us working on a Lie group, since we are working within an orthonormal basis, then the sectional curvature is given by:

$$\kappa(U,V) = \langle R(U,V)U,V\rangle$$

Theorem 4 If G is equipped with a left-invariant metric, then for any orthonormal basis on $\mathfrak{g}$, we have that the section curvature $\kappa(e_1,e_2)$ is given by

$$\kappa(e_1,e_2) = \tfrac{1}{2}(a_{12k}(-a_{12k}+a_{2k1}+a_{k12}) - \tfrac{1}{4}(a_{12k}-a_{2k1}+a_{k12})(a_{12k}+a_{2k1}-a_{k12}) - a_{k11}a_{k22})$$

where the constants a_{ijk} are the structure constants of $\mathfrak{g}$.

Proof. This is simply obtained by the first formula in the section and plugging it into definition for κ. □

In certain specific cases, this formula can simplify into a more useful form. If $ad(U)$ is self-adjoint for some $U \in \mathfrak{g}$, then

$$\kappa(U, V) \geq 0 \text{ for all } V \in \mathfrak{g}$$

Proof. Suppose U and V are orthonormal. Let $E_1 = U, E_2 = V, E_3, ..., E_n$ be an orthonormal basis. ad_U being skew-adjoint means that the array a_{ijk} is skew in the last two indices when $i = 1$. Then, the formula of Theorem 4 reduces to

$$\kappa(E_1, E_2) = \frac{1}{4}\sum_{l=1}^{n}(a_{2l1})^2$$

Therefore, $\kappa(E_1, E_2) \geq 0$. □

Furthermore, this is identically zero if U is orthogonal to the Lie span of v.

On an algebra, we define the center, $Z(\mathfrak{g})$ to be the collection of vectors that commute with all other elements. In the case of a Lie algebra, this means that a vector U is in the center if $[U, V] = 0$ for every $V \in \mathfrak{g}$. If V is in the center of $\mathfrak{g}$, then we also have $\kappa(U, V) \geq 0$ for all $V \in \mathfrak{g}$.

Proof. Let U be in the center. Then $Ad_U = 0$ and the zero transformation is skew-adjoint. So this follows from the above. □

5.2 Bi-invariant Case

Recall that for the bi-invariant case, we had the Levi-Civita connection as a simple formulation: $\nabla_X Y = \frac{1}{2}[X, Y]$.

From this we can also deduce a simple expression for the Riemannian curvature tensor. $R(U, V) = \frac{1}{4}ad_{[U,V]}$. Equivalently, we can say:

$$R(U, V)W = \frac{1}{4}[[U, V], W]$$

Additionally, the sectional curvature is given by:

$$\kappa(U, V) = \frac{1}{4}\langle [U, V], [U, V]\rangle \text{ for } U, V \text{ orthonormal}$$

Consequently, this means $\kappa(U, V) = 0$ if and only if $[U, V] = 0$.

5.3 Geodesics

The final Riemannian structure we discuss are geodesics. Generally, geodesics are auto-parallel curves, i.e., curves $\gamma : [0, 1] \to G$ such that $\nabla_{\gamma'}\gamma' = 0$. However, we must ask if there is a special characterization for our bi-invariant metric. Unsurprisingly, it turns out that the geodesics are just the integral curves of left-invariant vector fields on G.

The map defined by $I_g(h) = gh^{-1}g$ is also an isometry. This map has the special property that it reverses the orientation of geodesics. In other words, given a geodesic γ, then $I_g(\gamma(t)) = \gamma(-t)$. Geodesics through the identity e with respect to a vector field U are just the integral curves $\gamma(t) = \exp(tU)$, where this is the Lie exponential map. Since $\mathfrak{g} \cong T_eG$, this mapping corresponds the the Riemannian exponential map when viewing G as a Riemannian manifold.

However, in general the Riemannian exponential map and Lie exponential map are not necessarily the same as the Riemannian one depends on the metric. Consider that, given a vector field X, we can integrate to obtain a curve γ such that $\gamma'(t) = X(\gamma(t))$ for $t \in [0, 1]$.

Theorem 5 Let X be a left-invariant vector field with integral curve $\gamma(t)$ such that $\gamma(0) = e$ and $\gamma'(0) = X_0 \in T_eG$. Then $\gamma(t) = \exp(tX_0)$, where this is the Lie exponential map.

Proof. We have $\gamma'(t) = \exp(tX)X_0 = \gamma(t)X_0$. Therefore, $\gamma'(0) = eX_0 = x_0$ and $\gamma(0) = e$. Since X is left-invariant, then $X(\gamma(t)) = \gamma(t)X_0 = \gamma'(t)$. □

If we have the case of a bi-invariant metric, we do indeed see that the Riemannian and Lie exponential maps are the same. A result of this is:

Corollary 1 Let G be a connected Lie group with a bi-invariant metric. Then the Lie exponential map is surjective.

Proof. By theorem 5, since we have a bi-invariant metric, then the Riemannian and Lie exponential maps are the same. The Lie exponential map is defined on the entire tangent space for every $g \in G$, i.e. this means G is geodesically complete. Hence the same is so for the Riemannian exponential map. Then, by the Hopf-Rinow theorem, since the Riemannian exponential map is surjective, then so is the Lie exponential map. □

As a final note, if G is compact and connected, we obtain the same result since G being compact guarantees the existence of a bi-invariant metric.

6 References

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