SENSITIVITY ANALYSIS FOR OPTIMIZATION UNDER UNCERTAINTY

A. ALEXANDERIAN, J. DARGES, P. GREMAUD

1. Introduction

Let us consider the following optimization under uncertainty problem:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} F(\boldsymbol{x}) := \int_\Omega f(\boldsymbol{x},\boldsymbol{\theta})\mu(d\boldsymbol{\theta}) \tag{1}$$

Here $\boldsymbol{\theta}$ denotes realizations of a d-dimensional random vector that has distribution law μ. We let $\boldsymbol{x}^*$ denote the solution of the optimization problem, and ask the following question: how sensitive is $\boldsymbol{x}^*(\mu)$ to μ? Making this question precise and investigating it in a simple illustrative example is the goal of this preliminary study.

2. A concrete problem formulation

The problem described in the introduction can be made more concrete by (i) focusing on a specific set of distribution laws for $\boldsymbol{\theta}$; and, more importantly, (ii) specifying a suitable notion of sensitivity analysis in the present context. Let the law of $\boldsymbol{\theta}$ have density $\pi = \pi(\boldsymbol{\theta};\boldsymbol{\zeta})$, where $\boldsymbol{\zeta}$ is a vector of parameters characterizing π. For a given $\boldsymbol{\zeta}$, we consider the objective function

$$F(\boldsymbol{x};\boldsymbol{\zeta}) = \int f(\boldsymbol{x},\boldsymbol{\theta})\pi(\boldsymbol{\theta};\boldsymbol{\zeta}).$$

A minimizer $\boldsymbol{x}^*$ of F will be a function of $\boldsymbol{\zeta}$, $\boldsymbol{x}^* = \boldsymbol{x}^*(\boldsymbol{\zeta})$. We can then consider the sensitivity of $\boldsymbol{x}^*$ to the components of $\boldsymbol{\zeta}$.

3. Illustrative analytic example

In this section, we consider the optimization of the Rosenbrock function $f(\boldsymbol{x}) = (\theta_1 - \theta_2 x_1)^2 + \theta_3(x_2 - x_1^2)^2$ under uncertainty in the coefficients θ_1, θ_2, and θ_3.

In this brief study, we compare the solution of the corresponding (risk-neutral) optimization under uncertainty (OUU) problem versus the solution of the optimization problem with random and nominal choices of θ_i's. The eventual goal is to understand the sensitivity of the solution of the OUU problem to the distribution law of $\boldsymbol{\theta} = (\theta_1,\theta_2,\theta_3)^\top$, which is assumed to be a multivariate normal in the present study.

3.1. Model problem.

We consider the following "parameterized" Rosenbrock function:

$$f(\boldsymbol{x},\boldsymbol{\theta}) = (\theta_1 - \theta_2 x_1)^2 + \theta_3(x_2 - x_1^2)^2, \quad \boldsymbol{x}\in\mathbb{R}^3,\ \boldsymbol{\theta}\in\mathbb{R}^3. \tag{2}$$

For a fixed $\boldsymbol{\theta}$, and assuming $\theta_3 > 0$, this function attains its minimum at

$$\boldsymbol{x}^* = [\xi \quad \xi^2]^\top \quad \text{with} \quad \xi = \frac{\theta_1}{\theta_2}. \tag{3}$$

Note also that $f(\boldsymbol{x}^*,\boldsymbol{\theta}) = 0$. We make the following assumptions on the distribution of the random vector $\boldsymbol{\theta}$: we assume $[\theta_1 \quad \theta_2]^\top$ be a bivariate normal with mean $[m_1 \quad m_2]^\top$ and covariance matrix

$$\boldsymbol{C} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

We also assume that θ_3 is independent from θ_1 and θ_2 and is distributed uniformly in the interval $[99, 101]$. We consider the OUU problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} F(\boldsymbol{x}) := \int_{\mathbb{R}^3} f(\boldsymbol{x},\boldsymbol{\theta})\,\mu(d\boldsymbol{\theta}). \tag{4}$$

We can compute analytically that

$$F(\boldsymbol{x}) = (\sigma_2^2 + m_2^2)x_1^2 + m_3(x_2 - x_1^2)^2 - 2(\rho\sigma_1\sigma_2 + m_1m_2)x_1 + \sigma_1^2 + m_1^2$$

where $m_3 = 100$ is the mean of θ_3. The minimizer of this function is attained at

$$\boldsymbol{x}^* = [\xi \quad \xi^2]^\top, \quad \text{with} \quad \xi = \frac{\rho\sigma_1\sigma_2 + m_1m_2}{\sigma_2^2 + m_2^2}. \tag{5}$$

We can also compute

$$F(\boldsymbol{x}^*) = \frac{(\sigma_1^2 + m_1^2)(\sigma_2^2 + m_2^2) - (\rho\sigma_1\sigma_2 + m_1m_2)^2}{\sigma_2^2 + m_2^2} = \frac{\mathbb{E}(\theta_1^2)\mathbb{E}(\theta_2^2) - \mathbb{E}(\theta_1\theta_2)^2}{\mathbb{E}(\theta_2^2)}.$$

3.2. **Basic numerical illustrations.** First we specify the mean and covariance matrix of $\boldsymbol{\theta}$. We let $m_1 = 2$, $m_2 = 0.8$, $\sigma_1 = \sigma_2 = 0.1$, and $\rho = 0.7$. We compare the following:

- $\boldsymbol{x}^*_{\text{rand}}(\boldsymbol{\theta}) = \arg\min f(\boldsymbol{x},\boldsymbol{\theta})$.
- $\boldsymbol{x}^*_{\text{ouu}} = \arg\min \mathbb{E}_{\boldsymbol{\theta}}\big(f(\boldsymbol{x},\boldsymbol{\theta})\big)$.
- $\boldsymbol{x}^*_{\text{det}} = \arg\min f(\boldsymbol{x}, \mathbb{E}_{\boldsymbol{\theta}}(\boldsymbol{\theta}))$.

Note that $\boldsymbol{x}^*_{\text{rand}}$ is a random variable, whereas $\boldsymbol{x}^*_{\text{ouu}}$ and $\boldsymbol{x}^*_{\text{det}}$ are deterministic. Specifically, $\boldsymbol{x}^*_{\text{rand}}$ is defined according to (3), $\boldsymbol{x}^*_{\text{det}}$ is as in (3) with $\boldsymbol{\theta} = [m_1 \quad m_2 \quad m_3]^\top$, and $\boldsymbol{x}^*_{\text{ouu}}$ is as in (5). Since the second coordinate of each of these minimizers is determined based on the first one, we compare only the first coordinates of these different minimizers. The results are summarized in Figure 1.

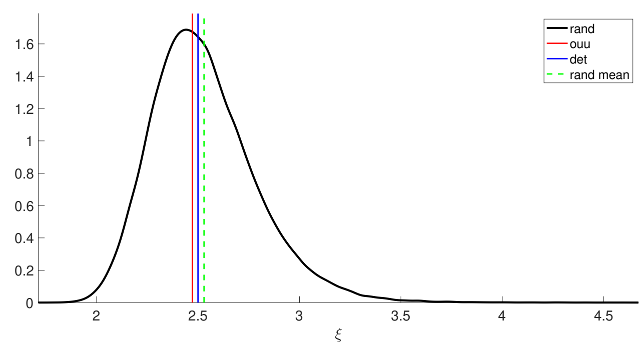

FIGURE 1. The first coordinate of $\boldsymbol{x}^*_{\text{rand}}$, $\boldsymbol{x}^*_{\text{ouu}}$ and $\boldsymbol{x}^*_{\text{det}}$, which we denote by ξ_{rand}, ξ_{ouu}, and ξ_{det}, respectively. The black curve depicts the probability density function of ξ_{rand}; the vertical lines show the location of (deterministic) ξ_{ouu} and ξ_{det}.

3.3. **Global sensitivity analysis.** To understand the sensitivity of the minimizer to the parameters determining the law of $\boldsymbol{\theta}$, we perform global sensitivity analysis (GSA). Since the minimizer is determined by the scalar term ξ defined in (5), it is sufficient to perform the analysis on this term. Note that $\xi = \xi(\zeta)$ with $\boldsymbol{\zeta} = [m_1 \quad m_2 \quad \sigma_1 \quad \sigma_2 \quad \rho]^\top$.

We model the uncertainty in the elements of $\boldsymbol{\zeta}$ by defining them as uniformly distributed random variables. Namely, for m_1, m_2, σ_1, and σ_2 we consider a 10% perturbation around the corresponding nominal values and let ρ be distributed uniformly in the interval $[-1, 1]$. We consider the following commonly used derivative based global sensitivity measures (DGSMs):

$$\nu_j = \mathbb{E}\Big\{\Big(\frac{\partial\xi}{\partial\zeta_j}\Big)^2\Big\}, \quad i = 1, \ldots, 5.$$

Here $\mathbb{E}$ denotes expectation over the law of $\boldsymbol{\zeta}$ and $n = 5$. It is important to note that here the parameters have different magnitudes and different ranges. A rational approach to standardize these indices is provided by the links between ν_j and the corresponding total Sobol' indices, T_j. Namely, we have

$$T_j \leq \frac{C_j}{V}\nu_j, \quad j = 1, \ldots, n,$$

where V denotes the variance of ξ and C_j are Poincare constants; for $\zeta_j \sim U(a_j, b_j)$, it is known that $C_j = (b_j - a_j)^2/\pi^2$. The bounds,

$$B_j = \frac{C_j}{V}\nu_j,$$

can be used to quantify parameter importance. Parameters with small corresponding B_j's can be considered unimportant.

In Figure 2 (left), we plot the bounds B_j, $j = 1, \ldots, 5$, for the entries of $\boldsymbol{\zeta}$. The results indicate that the minimizer of (4) is insensitive to σ_1, σ_2, and ρ. Essentially this means the uncertainty in $\boldsymbol{\theta}$, in this problem, does not have a significant influence on the minimizer. However, it is important to note that these results depend strongly on the nominal values of m_1, m_2, σ_1, and σ_2. We repeat the present numerical experiment with the following nominal values for the means and variances of θ_1 and θ_2:

$$\bar{m}_1 = 0.5, \, \bar{m}_2 = 0.35, \, \bar{\sigma}_1 = \bar{\sigma}_2 = 0.1.$$

The corresponding results are reported in Figure 2 (right). Note that in this case the DGSM-based bound for ρ has a nontrivial value indicating that this parameter is not necessarily unimportant. However, σ_1 and σ_2 are still unimportant.

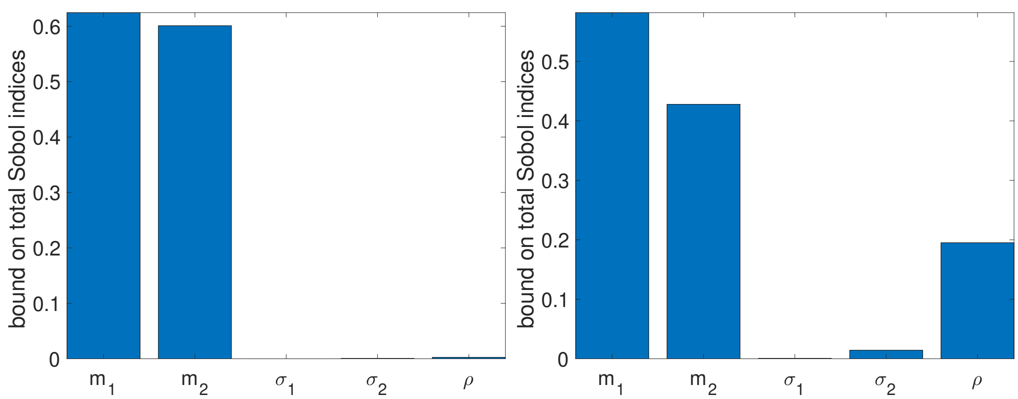

FIGURE 2. DGSM based bounds on total Sobol' indices.

We can evaluate the usefulness of these bounds by comparing their value to those of the numerically computed total Sobol' indices. The comparisons are showing in Figure 3 where the left and right figures correspond respectively to the left and right figures in Figure 2 .

In the example we are considering, this comparison suggests that significant about the Sobol' indices can be inferred from the DGSM bounds. The bounds not only mirror the magnitudes of the indices, they also relationships of magnitudes between different variables.

4. A practical note on implications of OUU: an optimal control perspective

The academic example consider here also provides some insight into performing OUU in practice. Note that for each fixed realization of $\boldsymbol{\theta}$ we can get an optimal $\boldsymbol{x}^*$ with optimal objective value of zero. However, $\boldsymbol{\theta}$ is uncertain. In practice, this can correspond to a control objective defined in terms of the output of a system that has some uncertain parameters in it and $\boldsymbol{x}$ is a control that we seek to optimize. Finding a control by minimizing $f(\boldsymbol{x}, \boldsymbol{\theta})$ for a fixed $\boldsymbol{\theta}$ will provide a control that

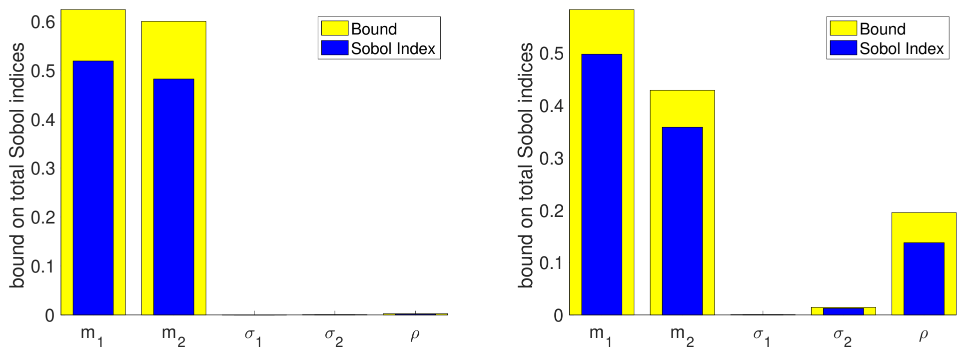

FIGURE 3. Comparison of DGSM bounds with the respective total Sobol' indices.

is optimal for that specific $\boldsymbol{\theta}$. However, in practice one does not know the value of $\boldsymbol{\theta}$ precisely—it is uncertain. A control computed with a fixed set of uncertain parameters can be grossly suboptimal for other possible realizations of $\boldsymbol{\theta}$. The idea of minimizing $F(\boldsymbol{x})$ is to obtain a control that is "good in average".